

# Asymptotics of Randomly Weighted $u$ - and $v$ -statistics: Application to Bootstrap

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## Abstract

This paper is mainly concerned with asymptotic studies of weighted bootstrap for  $u$ - and  $v$ -statistics. We derive the consistency of the weighted bootstrap  $u$ - and  $v$ -statistics, based on i.i.d. and non i.i.d. observations, from some more general results which we first establish for sums of randomly weighted arrays of random variables. Some of the results in this paper significantly extend some well-known results on consistency of  $u$ -statistics and also consistency of sums of arrays of random variables. We also employ a new approach to conditioning to derive a conditional CLT for weighted bootstrap  $u$ - and  $v$ -statistics, assuming the same conditions as the classical central limit theorems for regular  $u$ - and  $v$ -statistics.

Keywords: Conditional Central Limit Theorems, Laws of Large Numbers, Multinomial distribution,  $u$ - and  $v$ -statistics, Randomly Weighted  $u$ - and  $v$ -statistics, Weighted Arrays of random variables, Weighted Bootstrap

## 1 Introduction

The main purpose of this study is to investigate the validity of bootstrap  $u$ - and  $v$ -statistics resulting for the so-called  $m$ -out of- $n$  scheme of bootstrap.

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The use of this scheme results in having a randomly weighted form of the original  $u$ ,  $v$ -statistics. This, in turn, exhibits an explicit connection between the original  $u$ ,  $v$ -statistics and their bootstrap version. Moreover, it clearly identifies the two types of variations involved in a bootstrap problem for statistics of the form of partial sums in general and for  $u$ ,  $v$ -statistics in particular. In this exposition, the investigation of bootstrap  $u$ ,  $v$ -statistics includes the consistency and convergence in distribution of these bootstrap statistics. While the results for investigating the convergence in distribution for bootstrap is done only for  $u$ ,  $v$ -statistics, based on i.i.d. observations, when  $m$ -out of- $n$  scheme is used, the results in this paper on the consistency are provided in a more general way. In fact, the results on the consistency is restricted to neither  $u$ ,  $v$ -statistics nor to i.i.d. observations or bootstrap. The results on consistency are dealt with for randomly weighted arrays of random variables. These results then are used to derive consistency for randomly weighted  $u$ -statistics as well as for bootstrap  $u$ ,  $v$ -statistics based on i.i.d. observations and observations with absolute regularity property (cf. Theorems 3.1 and 3.2, respectively). Some of the results in this work extend well-known results on the strong law of large numbers for arrays of random variables and also for  $u$ -statistics (cf. Theorem 2.3 and Remark 2.4), and some of them shed light on the consistency of bootstrap when the original sample can stay finite (cf. Theorems 2.2 and 3.2).

The conditions assumed for the results in this paper are the same as the ones required for the classical central limit theorems (CLTs) and classical strong and weak laws of large numbers in the non-weighted case. In other words, there are no further restrictions imposed on the observations.

The material in this paper is organized as follows. In Section 2 laws of large numbers are provided for randomly weighted arrays of random variables (observations). The presentation of these results is so that they can be used in establishing the consistency of bootstrap  $u$ ,  $v$ -statistics which is provided in Section 3. Section 4 is devoted to establishing conditional (given the weights) central limit theorems for  $u$ ,  $v$ -statistics. In Section 5, remarks are made on the validity of the results for higher dimensional arrays of random variables and  $u$ ,  $v$ -statistics of order greater than 2. The proofs are given in Section 6.

## 2 Laws of large numbers for randomly weighted arrays of random variables

Consistency of bootstrap mean as randomly weighted sums was pioneered by Athreya [4] and followed by S. Csörgő [11] and Arenal-Gutiérrez *et al.* [3]. Since then the problem has received a great deal of attention from researchers. Of the contributions to the field, we specifically mention the two results, respectively due Arenal-Gutiérrez *et al.* [3], and Rosalsky and Sreehari [17], as the former influenced Theorem 3.2 and the latter motivated Theorem 2.1 of this exposition.

In this paper we consider two sequences of possibly double triangular (with respect to  $m$  and  $n$ ) arrays of random variables,  $\{X_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  and  $\{\varepsilon_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$ ,  $n, m \geq 1$ , which are defined on the same probability space  $(\Omega_{X,\varepsilon}, \mathfrak{F}_{X,\varepsilon}, P_{X,\varepsilon})$ . Also, by  $(\Omega_X, \mathfrak{F}_X, P_X)$  and  $(\Omega_\varepsilon, \mathfrak{F}_\varepsilon, P_\varepsilon)$  we denote the marginal probability spaces of the  $X^{(n,m)}$ s and  $\varepsilon^{(n,m)}$ s, respectively. We shall, often, refer to the  $X^{(n,m)}$ s as the observations (data) and  $\varepsilon^{(n,m)}$ s are referred to as the weights. We shall investigate the large sample behavior of the randomly weighted sums  $\sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)}$ ,  $n, m \geq 1$ . The observations and the weights are so that they can be employed in studying the  $m$ -out of- $n$  scheme of bootstrap for  $u$ -statistics which is to be discussed in Sections 3 and 4. We note in passing that when the observations and the weights are independent, then their joint probability space can of course be defined as the direct product probability space  $(\Omega_X \times \Omega_\varepsilon, \mathfrak{F}_X \otimes \mathfrak{F}_\varepsilon, P_X \times P_\varepsilon)$  of their marginals.

In this section we present some strong and weak consistency results for sums of randomly weighted arrays of random variables.

Except for Theorem 2.1 below and its application to bootstrap  $u$ -statistics in Theorem 3.1 of the next section, the method of conditioning plays an important role in the establishment of the results in this exposition. More precisely, employing hierarchical arguments, we derive our results via conditioning on the weights  $\varepsilon^{(n,m)}$  in some stochastic way with respect to  $P_\varepsilon$ . The latter results, in turn, can be extended to unconditional ones in terms of the joint probability measure  $P_{X,\varepsilon}$ . Hence, we let  $P_{|\varepsilon}(\cdot)$  and  $E_{|\varepsilon}(\cdot)$ , respectively stand for the conditional probability and conditional expected value given the weights  $\varepsilon^{(n,m)}$ .

The following Theorem 2.1 is a strong law of large numbers for sums of randomly weighted arrays of random variables.

**Theorem 2.1.** Consider the two possibly (double) triangular arrays  $\{X_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  and  $\{\varepsilon_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$ ,  $n, m \geq 1$ , of random variables which are defined on the same probability space. Let the sequence of positive integers  $m$  be such that  $m = m(n) \rightarrow +\infty$ , as  $n \rightarrow +\infty$ . Also let  $\{c_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  be a possibly (double) triangular array of real numbers and  $\{a_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  be a possibly (double) triangular array of positive real numbers such that, for  $\delta > 0$ ,

$$(a) P_\varepsilon(\cup_{1 \leq i, j \leq n} |\varepsilon_{ij}^{(n,m)} - c_{ij}^{(n,m)}| > \delta a_{ij}^{(n,m)}, i.o.(n)) = 0, \quad (1)$$

and, as  $m(n), n \rightarrow +\infty$ ,

$$(b) \sum_{1 \leq i, j \leq n} a_{ij}^{(n,m)} |X_{ij}^{(n,m)}| < +\infty \text{ a.s.} - P_X,$$

$$(c) \sum_{1 \leq i, j \leq n} c_{ij}^{(n,m)} X_{ij}^{(n,m)} < +\infty \text{ a.s.} - P_X.$$

Then, (a), (b) and (c), as  $n \rightarrow \infty$ , imply that

$\sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)}$  converges a.s.- $P_{X,\varepsilon}$  to the same a.s.- $P_X$  limit as that of  $\sum_{1 \leq i, j \leq n} c_{ij}^{(n,m)} X_{ij}^{(n,m)}$ .

**Remark 2.1.** In many cases it is natural for the numerical sequence  $c_{i,j}^{(n,m)}$  to be taken to be the mean of the weights, when it exists and is finite. In other words, for each,  $1 \leq i, j \leq n$ ,  $c_{i,j}^{(n,m)} = E_\varepsilon \varepsilon_{ij}^{(n,m)}$ ,  $n, m \geq 1$ . In this case, in view of the first Borel-Cantelli Lemma and Chebyshev inequality, perhaps the first and most natural way to investigate (1), when the weights have finite second moments, is to check if the following holds true:

$$\sum_{n=1}^{\infty} \sum_{1 \leq i, j \leq n} \frac{\text{Var}(\varepsilon_{ij}^{(n,m)})}{(a_{ij}^{(n,m)})^2} < +\infty.$$

This would amount to a generalization of Remark 1 of [17] to the case of arrays of random variables. However, in this exposition, when studying bootstrap  $u$ - and  $v$ -statistics in Section 3, we shall use Bernstein's inequality to investigate (1), as it gives sharper bounds with less computation when the weights are products of multinomially distributed random variables.

Theorem 2.1 above was motivated by, and it generalizes, Theorem 1 of Rosalsky and Sreehari [17] to arrays of random variables so that it can be used in studying the validity of the so called  $m$ -out of- $n$  method of bootstrap  $u$ -statistics in Section 3. It is obvious that when dealing with regular triangular arrays of random variables then, on taking  $\varepsilon^{(n,m)} = \varepsilon^{(n)}$  and  $X^{(n,m)} = X^{(n)}$ , Theorem 2.1 is true. It also continues to hold true for non-triangular observations and arrays.

The following Theorem 2.2 assumes that the observations have finite means and it concerns the randomly weighted sums of the form

$$\sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)}$$

of possibly (double) triangular arrays of observations  $\{X_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  and possibly (double) triangular arrays of random weights  $\{\varepsilon_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$ ,  $n, m \geq 1$ . The key difference between Theorem 2.1 above and the next result is that it does not necessarily require that both  $n, m \rightarrow +\infty$ . In fact it can be true when both, or either one of  $m$  and  $n$ , approach  $+\infty$ . As a consequence of this, it leads to an interesting application to bootstrap  $u$ ,  $v$ -statistics, via the  $m$ -out of- $n$  scheme, when the number of observations  $n$  is fixed and only the bootstrap sample size  $m$  approaches  $+\infty$ . The latter result is presented in Theorem 3.2 of the next section.

**Theorem 2.2.** *With the positive integers  $n, m \geq 1$ , let  $\{X_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$  and  $\{\varepsilon_{ij}^{(n,m)}; 1 \leq i, j \leq n\}$ , which are independent from each other, be possibly (double) triangular arrays of random variables such that, for  $1 \leq i, j \leq n$ ,  $E_X |X_{ij}^{(n,m)}| \leq c_{ij}^{(n,m)}$ , where  $c_{ij}^{(n,m)} > 0$ . Consider the following four statements:*

- (i)  $n$  is fixed and  $m \rightarrow +\infty$ ,
- (ii)  $m$  is fixed and  $n \rightarrow +\infty$ ,
- (iii)  $n, m \rightarrow +\infty$  such that  $m = m(n)$  is an increasing function of  $n$ ,

$$(iv) \sum_{1 \leq i, j \leq n} |\varepsilon_{ij}^{(n,m)}| c_{ij}^{(n,m)} = o(1) \text{ a.s.} - P_\varepsilon.$$

Then, either (i) and (iv), or (ii) and (iv), or (iii) and (iv) suffice to have

$$P_{X|\varepsilon} \left( \left| \sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)} \right| > \delta \right) \rightarrow 0 \text{ a.s.} - P_\varepsilon, \quad (2)$$

for any  $\delta > 0$ .

Also, when (iv) holds true in probability- $P_\varepsilon$  to begin with, then so is the conclusion (2).

**Remark 2.2.** Observe that when the observations are non-triangular, then naturally for the bounds we have  $c_{ij}^{(n,m)} = c_{ij}$ . In this case if  $\sup_{1 \leq i, j \leq +\infty} c_{ij} < +\infty$ , then (iv) of Theorem 2.2 can be replaced by

$$\sum_{1 \leq i, j \leq n} |\varepsilon_{ij}^{(n,m)}| = o(1) \text{ a.s.} - P_\varepsilon.$$

**Remark 2.3.** Since  $P_{X|\varepsilon}(|\sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)}| > \delta) \leq 1$ , by virtue of Lemma 1.2 in S. Csörgő and Rosalsky [10], conclusion (2) of Theorem 2.2, implies that

$$\sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)} \rightarrow 0, \text{ in probability} - P_{X,\varepsilon}.$$

The following result establishes a strong law of large numbers for sums of randomly weighted arrays of random variables when neither the observations nor the random random weights are triangular while both are assumed to be symmetric. This result is a generalization of the strong law of large numbers for  $u$ -statistics due to Berk [5] which was proven in view of martingale property of  $u$ -statistics. Some ideas were borrowed from the latter paper and adapted accordingly.

**Theorem 2.3.** Let  $X_{ij}$  and  $\varepsilon_{ij}$ ,  $1 \leq i \neq j \leq n$ ,  $n \geq 1$ , be symmetric arrays of random variables which are defined on the same probability space. If, for all  $i, j$ ,  $\varepsilon_{ij} X_{ij}$  are identically distributed and  $E_{X,\varepsilon}|\varepsilon_{12} X_{12}| < +\infty$ , then, as  $n \rightarrow \infty$ ,

$$S_n := \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \varepsilon_{ij} X_{ij} \rightarrow E_{X,\varepsilon}(\varepsilon_{12} X_{12}) \text{ a.s.} - P_{X,\varepsilon}.$$

Trivially, when  $\varepsilon_{ij}$  and  $X_{ij}$  are uncorrelated with  $E_\varepsilon|\varepsilon_{ij}| < +\infty$ , and  $E_X|X_{ij}| < +\infty$ , then the limit above becomes  $E_\varepsilon(\varepsilon_{12})E_X(X_{12})$ .

**Remark 2.4.** Theorem 2.3 generalizes Berk's [5] strong law of large numbers for regular  $u$ -statistics based on i.i.d. observations to randomly weighted arrays of random variables that may be taken to be the summands of a (randomly) weighted  $u$ -statistic based on identically distributed but not necessarily independent observations.

Theorem 2.3 also generalizes Theorem 2 of Etemadi [12], on taking  $m = n$  in the latter. The latter result of Etemadi requires that  $\epsilon_{ij}X_{ij}$ s be pairwise independent and identically distributed random variables, with respect to  $P_{X,\epsilon}$ , and that  $E_{X,\epsilon}\{|\epsilon_{ij}X_{ij}| \log |\epsilon_{ij}X_{ij}|\} < +\infty$ . As it can be readily seen, Theorem 2.3, drops the independence as well as it requires only  $E_{X,\epsilon}|\epsilon_{ij}X_{ij}| < +\infty$ .

We note in passing that Theorems 2.1, 2.2 and 2.3 assume almost no conditions (such as requiring certain kind of dependencies, for example) on the relation between the weights and the observations.

### 3 Consistency of bootstrap $u$ - & $v$ -statistics

Studying the problem of consistency of the bootstrap usually consists of showing that the deviation of the statistic in hand and its bootstrap version, perhaps multiplied by some normalizing sequence, vanishes a.s. or in probability with respect to the joint probability of the observations and the bootstrap experiment. In large sample applications of the bootstrap, one use of such an approximation is establishing limiting distribution results for a bootstrap statistic in hand. As an example in our context, we refer to the consistency in relation (37) below, noting that  $G_{n,m}$  is a deviation between a  $u$ -statistic with the kernel  $h(X_1, X_2) - \tilde{h}(X_1) - \tilde{h}(X_2)$  and its weighted bootstrap version. The need for such consistencies has first become apparent in A(6) of Burke and Gombay [7]. Motivated by this, and in view of our results in the previous section, in this section we give some results on the problem of bootstrapped  $u$ - and  $v$ -statistics when the original sample is large.

In small sample theory, the problem of consistency of bootstrap, as explained right above Theorem 2.2, leads to interesting results, if one can show that by re-sampling repeatedly from a finite sample, the thus obtained sequence of bootstrap versions of a statistic in hand converges to the original statistic a.s. or in probability, with respect to the joint probability of the observations and the bootstrap experiment. The results which we developed in Section 2 enable us to make such conclusions for bootstrapped  $u$ - and  $v$ -statistics when the original sample is finite (cf. Theorems 2.2 and 3.2).

In this section, as well as in the next one, we shall study the so-called  $m$ -out of- $n$  scheme of bootstrap which we are now to detail as follows.

## *m*-out of-*n* scheme of bootstrap

Draw a sample of size  $m \geq 1$  with replacement from the set  $\{1, \dots, n\}$  of the indices of the original observations  $X_1, \dots, X_n$ . In this scheme a bootstrap sub-sample of size  $m$ , denoted by  $X_1^*, \dots, X_m^*$ , is drawn from the original sample  $X_1, \dots, X_n$ ,  $n \geq 1$ , independently of the original observations.

In this section we study the consistency of bootstrap  $u$ - and  $v$ -statistics via the scheme of  $m$ -out of- $n$ . The obtained results will be seen to be applicable to i.i.d. and some stationary observations.

A  $u$ -statistic of order 2 is defined by

$$U_n = \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} h(X_i, X_j), \quad (3)$$

where,  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a measurable function which is symmetric in its arguments, and is called the kernel. Many of the well known statistics, such as sample variance, deleted jackknife variance estimator, Fisher's  $k$ -statistic for estimation of cumulants, Kendal's  $\tau$ , Gini's mean difference, are examples of  $u$ -statistics.

When investigating bootstrapped  $u$ -statistics, it is quite natural to consider  $v$ -statistics as well. A  $v$ -statistic can be viewed as an extension of a  $u$ -statistic of the same order, as it is defined as follows.

$$V_n = \frac{1}{n^2} \sum_{1 \leq i, j \leq n} h(X_i, X_j). \quad (4)$$

$$= \frac{n-1}{n} U_n + \frac{1}{n^2} \sum_{i=1}^n h(X_i, X_i), \quad (5)$$

where  $U_n$  is the  $u$ -statistic as in (3). The major difference between the  $u$ -statistic  $U_n$  and the  $v$ -statistic  $V_n$  is that the latter one includes the diagonal terms  $h(X_i, X_i)$ ,  $1 \leq i \leq n$ , while the former one doesn't. When the kernel  $h$  is so that  $h(x, y) = \ell(x - y)$ , where  $\ell$  is a function, then the corresponding  $u$ - and  $v$ -statistics coincide up to the constant  $(n-1)/n$ . Even when this is not the case, in view of (5), under some regularity conditions,  $U_n$  and  $V_n$  will asymptotically coincide.

The reason that one should consider  $v$ -statistics when bootstrapping  $u$ -statistics using the scheme of  $m$ -out of- $n$  is that, as a result of re-sampling with replacement, when computing the bootstrap  $u$ -statistic, for  $1 \leq s \neq$



$t \leq m$ , we may for example have  $X_s^* = X_1$  and  $X_t^* = X_1$ . This event has probability  $1/n^2$  that vanishes as  $n$  increases to  $+\infty$ .

A bootstrap  $u$ -statistic  $U_{n,m}^*$  and bootstrap  $v$ -statistic  $V_{n,m}^*$  based on a bootstrap sub-sample  $X_1^*, \dots, X_m^*$  of  $X_1, \dots, X_n$ , resulting from the method of  $m$ -out of- $n$  of bootstrap, are respectively defined as follows.

$$\begin{aligned} U_{n,m}^* &:= \frac{\sum_{\substack{1 \leq s \neq t \leq m \\ X_s^* \text{ and } X_t^* \text{ are distinct}}} h(X_s^*, X_t^*)}{m(m-1)} \\ &= \frac{\sum_{1 \leq i \neq j \leq n} w_i^{(n)} w_j^{(n)} h(X_i, X_j)}{m(m-1)} \end{aligned}$$

and

$$\begin{aligned} V_{n,m}^* &:= \frac{\sum_{1 \leq s \neq t \leq m} h(X_s^*, X_t^*)}{m(m-1)} \\ &= \frac{\sum_{1 \leq i, j \leq n} w_i^{(n)} w_j^{(n)} h(X_i, X_j)}{m(m-1)}, \end{aligned} \tag{6}$$

where  $w_i^{(n)}$  is the number of times the index  $i$ ,  $1 \leq i \leq n$ , is chosen in the scheme of  $m$ -out of- $n$ . In view of our earlier definition of this scheme of bootstrap it is obvious that  $w_i^{(n)}$ s are independent from the original observations  $X_1, \dots, X_n$ . Also, it is easy to see that  $w_i^{(n)}$ ,  $1 \leq i \leq n$ , to which we shall refer to as bootstrap weights or simply weights, are so that  $\sum_{i=1}^n w_i^{(n)} = m$  and  $E_w(w_i^{(n)}/m) = 1/n$ ,  $1 \leq i \leq n$ . That is, for each  $n \geq 1$ , the bootstrap weights have multinomial distribution with size  $m$ . In other words,

$$(w_1^{(n)}, \dots, w_n^{(n)}) \stackrel{d}{=} \text{multinomial}(m; \frac{1}{n}, \dots, \frac{1}{n}).$$

**Remark 3.1.** To state our results for  $U_{n,m}^*$  and  $V_{n,m}^*$ , it is important to address the relation between the two bootstrap statistics. Simple calculations show that, rhyming with (5), we have

$$V_{n,m}^* = U_{n,m}^* + \sum_{i=1}^n \frac{(w_i^{(n)})^2}{m(m-1)} h(X_i, X_i). \tag{7}$$

In this section, for a variety of observations, independent or dependent, we shall show that, as  $n, m \rightarrow +\infty$ , and at some point when only  $m \rightarrow +\infty$ , the

deviation between the bootstrap  $u$ -statistic  $U_{n,m}^*$  and the associated original  $u$ - and  $v$ -statistics  $U_n$  and  $V_n$  goes to zero a.s., or in probability, with respect to the joint distribution of  $P_{X,w}$ .

For the following result we consider i.i.d. observations and, as both  $m, n \rightarrow +\infty$ , we present a Marcinkiewicz type law of large numbers for  $U_{n,m}^*$  and  $V_{n,m}^*$  when the kernel  $h$  has less than one moment. In fact Theorem 3.1, below, can be viewed as a bootstrap version of Theorem 1 of Giné and Zinn [13]. This result is a consequence of Theorem 2.1 and reads as follows.

**Theorem 3.1.** *Let  $X_1, \dots, X_n$  be the first  $n \geq 1$  terms of an infinite sequence of i.i.d. random variables.*

(a) *Assume that  $E_X |h(X_1, X_2)|^{2/d} < +\infty$ , where  $d > 2$ . Then, as  $m, n \rightarrow +\infty$  such that  $m = O(n^{d/(d-2)} \log^{\frac{d+2}{d-2}} n)$ ,*

$$m^{-d+2} U_{n,m}^* \rightarrow 0 \text{ a.s.} - P_{X,w}.$$

(b) *Assume that  $E_X |h(X_1, X_2)|^{2/d} < +\infty$  and  $E_X |h(X_1, X_1)|^{2/d} < +\infty$ , where  $d > 2$ . Then, as  $m, n \rightarrow +\infty$  such that  $m = O(n^{d/(d-2)} \log^{\frac{d+2}{d-2}} n)$ ,*

$$m^{-d+2} V_{n,m}^* \rightarrow 0 \text{ a.s.} - P_{X,w}.$$

When the first moments of the kernel  $h$  of  $U_n$  and  $V_n$ , which does not depend on the sample size  $n$ , are uniformly bounded in  $1 \leq i, j < +\infty$ , then the following Theorem 3.2 is a direct consequence to Theorem 2.2 and Remark 2.3.

**Theorem 3.2.** *Let  $X_1, \dots, X_n$ ,  $n \geq 1$ , which are the first  $n \geq 1$  terms of an infinite sequence of random variables, and the kernel  $h$  be such that*

$$\sup_{1 \leq i, j < +\infty} E_X |h(X_i, X_j)| < +\infty.$$

(a) *For arbitrary  $\delta > 0$ , when the original sample size  $n$  is fixed and the bootstrap sample size  $m \rightarrow +\infty$ , then*

$$P_{X|w} \left( \left| U_{n,m}^* - \frac{n-1}{n} U_n \right| > \delta \right) \rightarrow 0 \text{ a.s.} - P_w. \quad (8)$$

$$P_{X|w} \left( |V_{n,m}^* - V_n| > \delta \right) \rightarrow 0 \text{ in probability} - P_w. \quad (9)$$

Consequently, for fixed  $n$ , as  $m \rightarrow +\infty$

$$U_{n,m}^* \rightarrow \frac{n-1}{n} U_n \text{ in probability } - P_{X,w}. \quad (10)$$

$$V_{n,m}^* \rightarrow V_n \text{ in probability } - P_{X,w}. \quad (11)$$

(b) For arbitrary  $\delta > 0$ , when  $m \rightarrow +\infty$  as  $n \rightarrow +\infty$  in such a way that  $m/(n\sqrt{2\log n}) \rightarrow +\infty$ , then both (8) and (9) hold true a.s.  $- P_w$  and, consequently,

$$U_{n,m}^* - U_n \rightarrow 0 \text{ in probability } - P_{X,w} \quad (12)$$

$$V_{n,m}^* - V_n \rightarrow 0 \text{ in probability } - P_{X,w}. \quad (13)$$

Part (a) of the preceding theorem is closer in spirit to what bootstrap is expected to do and better suites the inference one would wish to make as a result of re-sampling repeatedly from a finite original sample.

The next two results are to demonstrate the application of Theorem 3.2 for two types of observations. We first consider i.i.d. observations and in Corollary 3.1 we establish the validity of bootstrap  $u$ -statistic  $U_{n,m}^*$ . Then, in Corollary 3.2, we establish the validity of the method of  $m$ -out of- $n$  bootstrap for  $U_{n,m}^*$  when the observations posses the property of absolute regularity.

**Corollary 3.1.** *Let  $X_1, \dots, X_n$ , which are the first  $n \geq 1$  terms of an infinite sequence of i.i.d. random variables, and the kernel  $h$  be such that  $E_X|h(X_1, X_2)| < +\infty$  and  $E_X|h(X_1, X_1)| < +\infty$ . We conclude (a) and (b) as follows.*

(a) *For arbitrary  $\delta > 0$ , when  $n$  is fixed, as  $m \rightarrow +\infty$ , (8) and (9) and their respective consequences (10) and (11) hold true.*

(b) *For arbitrary  $\delta > 0$ , when  $n, m \rightarrow +\infty$  in such a way that  $m/(n\sqrt{2\log n}) \rightarrow +\infty$ , then*

$$P_{X|w}(|U_{n,m}^* - E_X h(X_1, X_2)| > \delta) \rightarrow 0 \text{ a.s. } - P_w, \quad (14)$$

$$P_{X|w}(|V_{n,m}^* - E_X h(X_1, X_2)| > \delta) \rightarrow 0 \text{ a.s. } - P_w. \quad (15)$$

Consequently,

$$U_{n,m}^* \rightarrow E_X h(X_1, X_2) \text{ in probability } - P_{X,w}, \quad (16)$$

$$V_{n,m}^* \rightarrow E_X h(X_1, X_2) \text{ in probability} - P_{X,w}. \quad (17)$$

It is noteworthy that the diagonal terms  $h(X_i, X_i)$ ,  $1 \leq i < +\infty$ , have no influence on the limit in (15) and (17).

As another example, we mention that Theorem 3.2 can be used to derive the validity of the bootstrap version of the  $v$ -statistic  $V_n$ , using the  $m$ -out-of- $n$  scheme. For its use in the next result, we now consider observations with absolute regularity property. A strong law of large numbers for  $u$ -statistics based on this type of observations was established by Arcones [1]. In fact part (b) of the following Corollary 3.2 is the bootstrap version of part (i) of Theorem 1 of the latter paper.

To state our next result, we first define the mixing coefficient  $\beta(k)$  as follows. Consider a strictly stationary sequence of random variables  $X_1, X_2, \dots$ , and the two  $\sigma$ -fields  $\mathfrak{T}_1^s := \sigma(X_1, \dots, X_s)$  and  $\mathfrak{L}_s^\infty := \sigma(X_s, X_{s+1}, \dots)$ . Let  $\{A_i\}_{i=1}^I$  be a partition in  $\mathfrak{T}_1^s$  and  $\{B_j\}_{j=1}^J$  be a partition in  $\mathfrak{L}_{s+k}^\infty$ , and define

$$\beta(k) := \frac{1}{2} \sup \left\{ \sum_{i=1}^I \sum_{j=1}^J |P_X(A_i \cap B_j) - P_X(A_i)P_X(B_j)|; \{A_i\}_{i=1}^I, \{B_j\}_{j=1}^J \right\}.$$

**Corollary 3.2.** *Let  $X_1, \dots, X_n$  be the first  $n \geq 1$  terms of an infinite sequence of strictly stationary random variables.*

(a) *If the kernel  $h$  is so that  $\sup_{1 \leq i, j < +\infty} E_X |h(X_i, X_j)| < +\infty$ , then, for fixed  $n$  and arbitrary  $\delta > 0$ , as  $m \rightarrow +\infty$ , (8) and (9) and their respective consequences (10) and (11) hold true.*

(b) *If for some  $s > 2$ ,  $\sup_{1 \leq i, j < +\infty} E_X |h(X_i, X_j)|^s < +\infty$ , and  $\beta(n) \rightarrow 0$ , as  $n \rightarrow +\infty$ , then, for arbitrary  $\delta > 0$ , as  $n, m \rightarrow +\infty$  in such a way that  $m/(n\sqrt{2\log n}) \rightarrow +\infty$ ,*

$$P_{X|w} \left( \left| U_{n,m}^* - \frac{\sum_{1 \leq i \neq j \leq n} E_X h(X_i, X_j)}{n(n-1)} \right| > \delta \right) \rightarrow 0 \text{ a.s.} - P_w, \quad (18)$$

$$P_{X|w} \left( \left| V_{n,m}^* - \frac{\sum_{1 \leq i \neq j \leq n} E_X h(X_i, X_j)}{n(n-1)} \right| > \delta \right) \rightarrow 0 \text{ a.s.} - P_w. \quad (19)$$

Consequently,

$$U_{n,m}^* - \frac{\sum_{1 \leq i \neq j \leq n} E_X h(X_i, X_j)}{n(n-1)} \rightarrow 0 \text{ in probability} - P_{X,w}, \quad (20)$$

$$V_{n,m}^* - \frac{\sum_{1 \leq i \neq j \leq n} E_X h(X_i, X_j)}{n(n-1)} \rightarrow 0 \text{ in probability} - P_{X,w}. \quad (21)$$

## 4 Asymptotic normality of bootstrap $u$ -statistics

Conditional (given the observations) central limit theorems for bootstrap  $u$ -statistics was investigated by Arcones and Giné [2], and Helmers [14]. However, the closest result in nature to the results in this section is the paper by Wang and Jing [20], where they provide Edgeworth expansions for the conditional (given the observations) distribution of weighted bootstrap  $u$ -statistics based on i.i.d. observations, assuming that the third moment of the kernel  $h$  exists and is finite. In contrast, on assuming that the kernel  $h$  has a finite second moment, we employ the method of conditioning on the bootstrap weights, which was first introduced by Csörgő *et al.* [8], to derive a conditional limit theorem for both bootstrap  $u$  and  $v$ -statistics.

Unlike the case of consistency of  $u$ -statistics, the concept of degeneracy is quite important in studying the asymptotic distribution of  $u$ -statistics and their associated  $v$ -statistics. This is a result of employing the celebrated method of Hoeffding reduction (decomposition) that involves the use of the projections of the underlying  $u$ -statistic. The number of non-zero projections determine the degree of degeneracy of a  $u$ -statistic (cf. for example Serfling [18] or Borovskikh [6]). In our case we shall consider the  $u$ - and  $v$ -statistics  $U_n$  and  $V_n$  defined in (3) and (4) which are non-degenerate, i.e., for all  $i \geq 1$ ,

$$\tilde{h}(X_i) := E_X(h(X_i, X_j) - E_X h(X_i, X_j) | X_i) \neq 0 \text{ a.s.} - P_X,$$

where  $1 \leq j \neq i$ .

The concept of degeneracy is inherited by the  $u$ -statistic  $U_n$  from its kernel  $h$ . By this, and in a similar fashion, the concept of degeneracy can be extended to be used for *weighted  $u$ -statistics* as in Nasari [15]. Degenerate weighted  $u$ -statistics are to be used in our proofs.

Let  $T_{n,m}^*$  be the bootstrap version of an original statistic  $T_n$ , when the  $m$ -out of- $n$  scheme of bootstrap is used. The validity in distribution of the  $m$ -out of- $n$  method of bootstrap should *ideally* be investigated by directly showing that, for fixed sample size  $n$  and all  $t \in \mathbb{R}$ , as only  $m \rightarrow +\infty$ , one has

$$P_{X,w}(T_{n,m}^* \leq t) \rightarrow P_X(T_n \leq t). \quad (22)$$

This is of the same spirit as part (a) of Theorem 3.2, and its Corollaries 3.1 and 3.2 in the previous section. However, unlike the latter results, (22) cannot be directly investigated as such investigation would require knowledge of the sampling distribution of the statistic  $T_n$  for fixed  $n \geq 1$ , which usually is not the case, as in practice the sampling distribution is unknown. Therefore, (22) is usually established by showing the nearness of  $P_{X,w}(T_{n,m}^* \leq t)$  to the limiting distribution of the original statistic  $T_n$ , as  $m, n \rightarrow +\infty$ . Denoting the limiting distribution of properly normalized and (usually) centered  $T_n$  by  $F$ , in the literature, as  $n, m \rightarrow +\infty$ , the stronger conditional (given the sample) version

$$P(b_n(T_{n,m}^* - T_n) \leq t | X_1, \dots, X_n) \rightarrow F(t) \text{ a.s.} - P_X \text{ or in probability} - P_X \quad (23)$$

is what is shown to establish (22), where  $b_n$  is a normalizing sequence. For the Student  $t$ -statistic, when the limiting distribution  $F$  is standard normal, Csörgő *et al.* [8] established the validity of  $m$ -out of- $n$  scheme of bootstrap via the classical method of (23), as well as using a new method of conditioning on the weights, by showing that, as  $n, m \rightarrow +\infty$ ,

$$P_{X|w}(T_{n,m}^* \leq t) \rightarrow F(t) \text{ in probability} - P_w.$$

In this section, for i.i.d. observations, we shall establish the validity, in distribution, of the bootstrap  $u$ - and  $v$ -statistics by proving a similar result to the preceding relation, i.e., via conditioning on the bootstrap weights.

**Theorem 4.1.** *Let  $X_1, \dots, X_n$  be the first  $n \geq 1$  terms of an infinite sequence of i.i.d. random variables. Assume that  $U_n$  and  $V_n$  are non-degenerate.*

*(a) If  $E_X h^2(X_1, X_2) < +\infty$ , then, for all  $t \in \mathbb{R}$ , as  $m, n \rightarrow +\infty$  in such a way that  $m = o(n^2)$ ,*

$$P_{X|w}\left(\frac{U_{n,m}^* - U_n}{2 \hat{\sigma}_n \sqrt{\sum_{t=1}^n \left(\frac{w_t^{(n)}}{m} - \frac{1}{n}\right)^2}} \leq t\right) \rightarrow \Phi(t) \text{ in probability} - P_w \quad (24)$$

*(b) If  $E_X h^2(X_1, X_2), E_X h^2(X_1, X_1) < +\infty$ , then, for all  $t \in \mathbb{R}$ , as  $m, n \rightarrow +\infty$  in such a way that  $m = o(n^2)$ ,*

$$P_{X|w}\left(\frac{V_{n,m}^* - V_n}{2 \hat{\sigma}_n \sqrt{\sum_{t=1}^n \left(\frac{w_t^{(n)}}{m} - \frac{1}{n}\right)^2}} \leq t\right) \rightarrow \Phi(t) \text{ in probability} - P_w, \quad (25)$$

where  $\Phi$  is the distribution function of a standard normal random variable,

$$\hat{\sigma}_n^2 = n(n-1) \sum_{i=1}^n (U_{n-1}^i - U_n)^2, \quad (26)$$

and  $U_{n-1}^i$  is the jackknifed version of  $U_n$ , based on  $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n$ , defined as follows

$$U_{n-1}^i := \frac{1}{\binom{n-1}{2}} \sum_{\substack{1 \leq j_1 < j_2 \leq n \\ j_1, j_2 \neq i}} h(X_{j_1}, X_{j_2}).$$

**Remark 4.1.**  $\hat{\sigma}_n^2$  is the jackknife estimator of  $\sigma^2 := \text{Var}(\tilde{h}(X_1))$ . It was used by Csörgő et al. [9] for  $u$ -type processes when studying the changepoint problem via Studentization. The estimator  $\hat{\sigma}_n^2$  was generalized by Nasari [16] for processes of  $u$ -statistics of order greater than or equal to 2. It is noteworthy that  $\hat{\sigma}_n^2$  remains the right normalizing sequence for the weak convergence (and central limit theorem as a result) of  $u$ -statistics even when  $Eh^2(X_1, X_2) = +\infty$  (cf. Theorem 4 and its Corollary 1 of Nasari [16]).

We call attention to Theorem 4.1 being valid when bootstrap sub-samples of size  $m = n$ , or smaller than  $n$ , are drawn from the original sample.

**Remark 4.2.** In view of Lemma 1.2 in S. Csörgő and Rosalsky [10], we note that the respective unconditional versions of (24) and (25) continue to hold true, i.e., having  $P_{X,w}$  instead of  $P_{X|w}$  in their respective statements with  $\Phi$  as their limiting distribution.

## 5 Remarks on extending the results to $p$ -dimensional case

Theorems 2.1, 2.2 and 2.3 continue to hold true for sums of randomly weighted  $p$ -dimensional arrays of randomly weighted random variables, where  $p$  is a positive integer such that  $p \geq 2$ . Theorems 3.1, 3.2 and Corollaries 3.1 and 3.2 also hold true for  $u$ - and  $v$ -statistics of order  $p \geq 2$ . Mutatis mutandis, the respective proofs of these results in the  $p$ -dimensional case remain the same as in the 2-dimensional case of the present paper. As for Theorem 4.1, when dealing with  $u$ -statistics of order  $p \geq 2$ , the theorem remains valid

assuming the same conditions. The only change required is to change the constant 2 in the denominators of (24) and (25) to  $p$ . The proof of this theorem for non-degenerate  $u$ ,  $v$ -statistics of order  $p$  is in principle the same (cf. Appendix).

## 6 Proofs

### 6.1 Proof of Theorem 2.1

The proof of this theorem can be done by modifying the proof of Theorem 1 of [17] as follows:

$$\begin{aligned}
& \left| \sum_{1 \leq i, j \leq n} \varepsilon_{ij}^{(n,m)} X_{ij}^{(n,m)} - \sum_{1 \leq i, j \leq n} c_{ij}^{(n,m)} X_{ij}^{(n,m)} \right| \\
& \leq \sum_{1 \leq i, j \leq n} |\varepsilon_{ij}^{(n,m)} - c_{ij}^{(n,m)}| |X_{ij}^{(n,m)}| \\
& \leq \delta \sum_{1 \leq i, j \leq n} a_{ij}^{(n,m)} |X_{ij}^{(n,m)}| \rightarrow 0 \text{ a.s.} - P_X,
\end{aligned}$$

as  $n \rightarrow \infty$ ,  $\delta \rightarrow 0$ .  $\square$

### 6.2 Proof of Theorem 2.2

The proof of this theorem is relatively simple and we give the details only for the case when the convergence is a.s.  $-P_\varepsilon$ .

First consider the case when  $n$  is fixed and  $m \rightarrow +\infty$ , i.e., case (i), and, for  $\delta_1, \delta_2 > 0$ , write

$$\begin{aligned}
& P_w \left\{ \cap_{m \geq 1} \cup_{k \geq m} P_{X|\varepsilon} \left( \left| \sum_{1 \leq i, j \leq n} \varepsilon_{i,j}^{(n,k)} X_{i,j}^{(n,k)} \right| > \delta_1 \right) > \delta_2 \right\} \\
& \leq P_w \left\{ \cap_{m \geq 1} \cup_{k \geq m} \sum_{1 \leq i, j \leq n} |\varepsilon_{i,j}^{(n,k)}| E_X |X_{i,j}^{(n,k)}| > \delta_1 \delta_2 \right\} \\
& \leq P_w \left\{ \cap_{m \geq 1} \cup_{k \geq m} \sum_{1 \leq i, j \leq n} |\varepsilon_{i,j}^{(n,k)}| c_{i,j}^{(n,k)} > \delta_1 \delta_2 \right\} \\
& = 0.
\end{aligned}$$

The latter conclusion is true in view of assumption (iv), when (i) holds.



On exchanging  $m$  by  $n$  in the preceding argument, the proof of this theorem when (ii) holds will follow. Hence the details are omitted.

When (iii) holds,  $m = m(n) \rightarrow +\infty$ , as  $n$  does so, and with  $\delta_1, \delta_2 > 0$ , similarly to the proof for the case (i), we write

$$\begin{aligned}
& P_w \{ \cap_{n \geq 1} \cup_{k \geq n} P_{X|\varepsilon} (| \sum_{1 \leq i, j \leq k} \varepsilon_{i,j}^{(k,m(k))} X_{i,j}^{(k,m(k))} | > \delta_1) > \delta_2 \} \\
& \leq P_w \{ \cap_{n \geq 1} \cup_{k \geq n} \sum_{1 \leq i, j \leq k} | \varepsilon_{i,j}^{(k,m(k))} | E_X | X_{i,j}^{(k,m(k))} | > \delta_1 \delta_2 \} \\
& \leq P_w \{ \cap_{n \geq 1} \cup_{k \geq n} \sum_{1 \leq i, j \leq k} | \varepsilon_{i,j}^{(k,m(k))} | c_{i,j}^{(k,m(k))} > \delta_1 \delta_2 \} \\
& = 0.
\end{aligned}$$

Once again, the latter conclusion is true in view of assumption (iv), when (iii) holds. This also completes the proof of Theorem 2.2.  $\square$

### 6.3 Proof of Theorem 2.3

The proof of this theorem relies on the concept of symmetric (permutable) events, and is similar to the proof of the SLLN for  $U$ -statistics due Berk [5]. To prove this theorem, we first define the  $\sigma$ -field  $\mathfrak{F}_n$  as follows:

$$\mathfrak{F}_n = \sigma(S_n, S_{n+1}, \dots).$$

Note that, by definition,  $\mathfrak{F}_{n+1} \subset \mathfrak{F}_n$ , and observe that, since the  $\varepsilon_{ij} X_{ij}$ 's are identically distributed, for each  $n$  and each  $1 \leq i \neq j \leq n$  we have

$$\begin{aligned}
E(\varepsilon_{12} X_{12} | \mathfrak{F}_n) &= E(\varepsilon_{ij} X_{ij} | \mathfrak{F}_n) \text{ a.s.} - P_{X,\varepsilon} \\
&= E\left(\frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} \varepsilon_{ij} X_{ij} | \mathfrak{F}_n\right) \text{ a.s.} - P_{X,\varepsilon} \\
&= E(S_n | \mathfrak{F}_n) \text{ a.s.} - P_{X,\varepsilon} \\
&= S_n \text{ a.s.} - P_{X,\varepsilon}.
\end{aligned}$$

In view of the latter, and by Theorem 2.8.6 of [19], we now conclude that, as  $n \rightarrow \infty$ , we have

$$S_n = E(\varepsilon_{12} X_{12} | \mathfrak{F}_n) \longrightarrow E(\varepsilon_{12} X_{12} | \cap_{2 \leq n < \infty} \mathfrak{F}_n) \text{ a.s.} - P_{X,\varepsilon}.$$

Now, due to symmetry (permutability) of the tail events (cf. [19]), it follows by the Hewitt-Savage 0-1 law (cf. for example Stout [19] and Theorem 2.12.4 therein) that  $E(\epsilon_{12}X_{12} | \cap_{2 \leq n < \infty} \mathfrak{F}_n)$  is a constant a.s.- $P_{X,\epsilon}$ . This, in turn means that, as  $n \rightarrow +\infty$ ,

$$S_n \rightarrow E_{X,\epsilon}(\epsilon_{12}X_{12}) \text{ a.s. } - P_{X,\epsilon}.$$

The proof of Theorem 2.3 is now complete.  $\square$

## 6.4 Proof of Theorem 3.1

To prove this result, by virtue of (5) we first observe that

$$\begin{aligned} m^{-d+2}U_{n,m}^* &= \sum_{1 \leq i \neq j \leq n} \frac{w_i^{(n)}w_j^{(n)}}{m^{d-1}(m-1)} h(X_i, X_j) \\ &+ \sum_{i=1}^n \frac{w_i^{(n)^2}}{m^{d-1}(m-1)} h(X_i, X_i). \end{aligned} \quad (27)$$

Also, from Theorem 1 of Giné and Zinn [13], we know that, as  $n \rightarrow +\infty$ ,  $n^{-d} \sum_{1 \leq i \neq j \leq n} h(X_i, X_j) \rightarrow 0$ . Also, observe that, for  $\delta > 0$ ,

$$\begin{aligned} P_X \left( \frac{\left| \sum_{i=1}^n h(X_i, X_i) \right|}{n^d \log^2 n} > \delta \right) &\leq \delta^{-2/d} \frac{n}{n^2 \log^2 n} E_X \left| h(X_1, X_1) \right|^{2/d} \\ &= \delta^{-2/d} \frac{E_X \left| h(X_1, X_1) \right|^{2/d}}{n \log^2 n}. \end{aligned}$$

Now the proof of Theorem 3.1 results from an application of Theorem 2.1 to both terms on the R.H.S. of the equality (27), on taking

$$c_{ij}^{(n,m)} = E_w \left( \frac{w_i^{(n)}w_j^{(n)}}{m^{d-1}(m-1)} \right) = \begin{cases} n^{-2}m^{-d+2}, & i \neq j; \\ \frac{1-1/n}{nm^{d-2}(m-1)} + \frac{1}{n^2m^{d-3}(m-1)}, & i = j \end{cases}$$

and

$$a_{ij}^{(n,m)} = \begin{cases} n^{-d}, & i \neq j; \\ n^{-d} \log^{-d} n, & i = j. \end{cases}$$

Part (a) follows if, as  $n, m \rightarrow +\infty$  such that  $m = O(n^{d/(d-2)} \log^{\frac{d+2}{d-2}} n)$ , for  $\delta > 0$ , we show that

$$\sum_{n \geq 1} n(n-1)P_w \left( \left| \frac{w_1^{(n)}w_2^{(n)}}{m^{d-1}(m-1)} - \frac{1}{n^2m^{d-2}} \right| > \delta n^{-d} \right) < +\infty. \quad (28)$$

In view of (27), part (b) will follow if, in addition to (28), we prove that

$$\sum_{n \geq 1} n P_w \left( \left| \frac{(w_1^{(n)})^2}{m^{d-1}(m-1)} - \left( \frac{1-1/n}{nm^{d-2}(m-1)} + \frac{1}{n^2 m^{d-3}(m-1)} \right) \right| > \frac{\delta}{n^d \log^d n} \right) < +\infty. \quad (29)$$

To establish (28), we first note that without, loss of generality, we may assume that, for each  $1 \leq i \neq j \leq n$  and  $n \geq 1$ ,  $w_i^{(n)} w_j^{(n)} \stackrel{d}{=} \text{Binomial}(m(m-1), 1/n^2)$ . This, in turn, enables us to use Bernstein's inequality to estimate

$$\begin{aligned} & n(n-1) P_w \left( \left| \frac{w_1^{(n)} w_2^{(n)}}{m^{d-1}(m-1)} - \frac{1}{n^2 m^{d-2}} \right| > \delta n^{-d} \right) \\ & \leq n(n-1) \exp \left\{ -m(m-1) \frac{\delta^2 m^{2d-4}/n^{2d}}{2(\frac{1}{n^2} + \delta m^{d-2}/n^d)} \right\} \\ & \sim n^2 \exp \left\{ -\frac{m^d}{n^d} \frac{\delta^2}{2(n^{d-2}/m^{d-2} + \delta)} \right\}. \end{aligned}$$

The last term above is a general term of a convergent series when  $n, m \rightarrow +\infty$  such that  $m = O(n^{d/(d-2)} \log^{\frac{d+2}{d-2}} n)$ . This completes the proof of (28) and hence that of part (a).

To prove (29) we use Markov's inequality to estimate

$$\begin{aligned} & n P_w \left( \left| \frac{(w_1^{(n)})^2}{m^{d-1}(m-1)} - \left( \frac{1-1/n}{nm^{d-2}(m-1)} + \frac{1}{n^2 m^{d-3}(m-1)} \right) \right| > \frac{\delta}{n^d \log^d n} \right) \\ & \leq 2\delta^{-1} n^{d+1} \log^d n \left( \frac{1-1/n}{nm^{d-2}(m-1)} + \frac{1}{n^2 m^{d-3}(m-1)} \right) \\ & \sim \frac{2n^d \log^d n}{m^{d-1}} + \frac{2n^{d-1} \log^d n}{m^{d-2}}. \end{aligned}$$

The last term above is a general term of a convergent series when  $n, m \rightarrow +\infty$  such that  $m = O(n^{d/(d-2)} \log^{\frac{d+2}{d-2}} n)$ . Consequently, we conclude (29) and, in view of having also (28), the proof of part (b) is complete. This also completes the proof of Theorem 3.1.  $\square$

## 6.5 Proof of Theorem 3.2

On taking  $c_{ij}^{(n,m)} = c_{ij} = E_X |h(X_i, X_j)|$  and noting that here  $\sup_{1 \leq i, j < +\infty} c_{ij} < +\infty$ , the proof of (8) will follow from Theorem 2.2 and Remark 2.2 if we

show that, as  $m \rightarrow +\infty$ ,

$$\sum_{1 \leq i \leq j \leq n} \left| \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n^2} \right| = o(1) \text{ a.s.} - P_w. \quad (30)$$

In view of Remark 3.1, as  $m \rightarrow +\infty$ , (9) results from the preceding (30) and the following (31):

$$\sum_{i=1}^n \left| \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n^2} \right| = o_{P_w}(1). \quad (31)$$

To prove (30), we use Bernstein's inequality along the lines of the following argument to write

$$\begin{aligned} & P_w \left( \sum_{1 \leq i \neq j \leq n} \left| \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n^2} \right| > \delta \right) \\ & \leq n(n-1) P_w \left( \left| \frac{w_1^{(n)} w_2^{(n)}}{m(m-1)} - \frac{1}{n^2} \right| > \frac{\delta}{n(n-1)} \right) \\ & \leq n(n-1) \exp \left\{ \frac{-m(m-1)}{n(n-1)} \cdot \frac{\delta^2}{2(\frac{n(n-1)}{n^2} + \delta)} \right\}. \end{aligned} \quad (32)$$

Observe now that for the latter upper bound in (32), as  $n$  is fixed, we have

$$\sum_{m \geq 1} \exp \left\{ \frac{-m(m-1)}{n(n-1)} \cdot \frac{\delta^2}{2(\frac{n(n-1)}{n^2} + \delta)} \right\} < +\infty.$$

This completes the proof of (30).

To establish (31), we begin with an application of Chebyshev inequality to write

$$\begin{aligned} P_w \left( \sum_{i=1}^n \left| \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n^2} \right| > \delta \right) & \leq \delta^{-2} n^2 E_w \left( \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n^2} \right)^2 \\ & \sim \delta^{-2} n^2 \left( \frac{1}{nm^3} + \frac{7}{n^2 m^2} + \frac{6}{mn^3} - \frac{2(1 - \frac{1}{n})}{n^3 m} \right) \\ & \leq \delta^{-2} n^2 \left( \frac{1}{nm^3} + \frac{7}{n^2 m^2} + \frac{6}{mn^3} \right). \end{aligned} \quad (33)$$

The preceding relation approaches zero as  $m \rightarrow +\infty$ . This completes the proof of (31). Noting now that, in view of Lemma 1.2 in S. Csörgő and Rosalsky [10], (10) and (11) follow from (8) and (9), respectively, as  $m \rightarrow +\infty$ , completes the proof of part (a) of Theorem 3.2.

To prove part (b) of this theorem, it suffices to observe that the respective upper bounds in (32) and (33) are both general terms of a convergent series in  $n$ , when the bootstrap sample size  $m$  is so that  $m/(n\sqrt{2\log n}) \rightarrow +\infty$ . In other words,

$$\sum_{n \geq 1} n(n-1) \exp \left\{ \frac{-m(m-1)}{n(n-1)} \cdot \frac{\delta^2}{2(\frac{n(n-1)}{n^2} + \delta)} \right\} < +\infty$$

and

$$\sum_{n \geq 1} n^2 \left( \frac{1}{nm^3} + \frac{7}{n^2 m^2} + \frac{6}{mn^3} \right) < +\infty.$$

Once again, the relations (10) and (11) result from Lemma 1.2 in S. Csörgő and Rosalsky [10], and this completes the proof of part (b) and that of Theorem 3.2.  $\square$

## 6.6 Proof of Corollary 3.1

Part (a) is a trivial consequence to Theorem 3.2. Here we only have to argue part(b).

From the strong law of large numbers for  $u$ -statistics based on i.i.d. observations we know that, as  $n \rightarrow +\infty$ ,  $U_n \rightarrow E_X h(X_1, X_2)$  a.s.- $P_X$  (cf. for example Serfling [18]). This combined with (8) imply (14) and its consequence (16).

By virtue of (5), the proof of (15) and its consequence (17) follow from the fact that  $X_i$ s are i.i.d. random variables and  $E_X |h(X_1, X_1)| < +\infty$  and as a result, as  $n \rightarrow +\infty$ ,

$$\frac{1}{n^2} \sum_{i=1}^n h(X_i, X_i) \rightarrow 0 \text{ a.s. } - P_X.$$

## 6.7 Proof of Corollary 3.2

Once again Part (a) is a trivial consequence of Theorem 3.2.

From Theorem 1 of Arcones [1] we know that, as  $n \rightarrow +\infty$ ,  $U_n - \frac{1}{n(n-1)} \sum_{1 \leq i \neq j \leq n} E_X h(X_i, X_j) \rightarrow 0$  a.s.- $P_X$ . This combined with (8) imply (18) and its consequence (20).

In view of (5), to prove (19) and its consequence (21), it suffices to show that

$$\frac{1}{n^2} \sum_{1 \leq i \leq n} h(X_i, X_i) \rightarrow 0 \text{ a.s.} - P_X.$$

To establish the latter, for  $\delta > 0$ , we use the Chebyshev inequality to write

$$\begin{aligned} P_w \left( \left| \frac{1}{n^2} \sum_{1 \leq i \leq n} h(X_i, X_i) \right| > \delta \right) &\leq \delta^{-2} n^{-4} \sum_{1 \leq i, j \leq n} E_X |h(X_i, X_i) h(X_j, X_j)| \\ &\leq \delta^{-2} n^{-4} \sum_{1 \leq i, j \leq n} E_X^{1/2} h^2(X_i, X_i) E_X^{1/2} h^2(X_j, X_j) \\ &\leq \delta^{-2} n^{-2} \left( \sup_{1 \leq i < +\infty} E_X^{1/2} h^2(X_i, X_i) \right)^2. \end{aligned}$$

Now the proof of Corollary 3.2 is complete.  $\square$

## 6.8 proof of Theorem 4.1

By virtue of Remark 3 of Nasari [16] we may replace  $\hat{\sigma}_n^2$  by  $\sigma^2 = E_X(\tilde{h})^2 \leq E_X h^2(X_1, X_2) < +\infty$ . This is so since, as  $n \rightarrow +\infty$ ,  $\hat{\sigma}_n^2 \rightarrow \sigma^2$  in probability- $P_X$ . Therefore, to prove part (a) of this theorem, it suffices to show that

$$P_{X|w} \left( \frac{U_{n,m}^* - U_n}{2\sigma \sqrt{\sum_{t=1}^n \left( \frac{w_t^{(n)}}{m} - \frac{1}{n} \right)^2}} \leq t \right) \rightarrow \Phi(t) \text{ in probability} - P_w \quad (34)$$

Observe now that  $U_{n,m}^* - U_n$  can be written as a weighted  $u$ -statistic as follows:

$$U_{n,m}^* - U_n = \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) h(X_i, X_j). \quad (35)$$

In view of the preceding relation, the fact that  $\sum_{1 \leq i \neq j \leq n} \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} = 1$  allows us to assume without loss of generality that  $E_X h(X_1, X_2) = 0$ . In other words, the difference  $U_{n,m}^* - U_n$  does not feel the theoretical mean of the original  $u$ -statistic  $U_n$ .

We now employ a Hoeffding type reduction for the weighted  $u$ -statistic  $U_{n,m}^* - U_n$ , as in (35), and write

$$\begin{aligned}
U_{n,m}^* - U_n &= \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) (h(X_i, X_j) - \tilde{h}(X_i) - \tilde{h}(X_j)) \\
&+ 2 \sum_{1 \leq i \neq j \leq n} \left( \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) \tilde{h}(X_i) \\
&=: G_{n,m} + H_{n,m}.
\end{aligned} \tag{36}$$

The proof of part (a) follows if we show that, as  $n, m \rightarrow +\infty$  such that  $m = o(n^2)$ , then, for  $\delta_1, \delta_2 > 0$ ,

$$P_w \left\{ P_{X|w} \left( \frac{|G_{n,m}|}{\sqrt{\sum_{t=1}^n \left( \frac{w_t^{(n)}}{m} - \frac{1}{n} \right)^2}} > \delta_1 \right) > \delta_2 \right\} = o(1), \tag{37}$$

and for all  $t \in \mathbb{R}$ ,

$$P_{X|w} \left( \frac{H_{n,m}}{\sigma \sqrt{\sum_{t=1}^n \left( \frac{w_t^{(n)}}{m} - \frac{1}{n} \right)^2}} \leq t \right) \rightarrow \Phi(t) \text{ in probability} - P_w. \tag{38}$$

We are now to prove (37). To do so we first show that as  $n, m \rightarrow +\infty$  such that  $m = o(n^2)$ , for  $\delta > 0$

$$P_w \left( \left| \frac{m}{(1 - \frac{1}{n})} \sum_{t=1}^n \left( \frac{w_t^{(n)}}{m} - \frac{1}{n} \right)^2 - 1 \right| > \delta \right) \rightarrow 0. \tag{39}$$

First note that for each  $i$ ,  $1 \leq i \leq n$ ,

$$E_w \left( \frac{w_t^{(n)}}{m} - \frac{1}{n} \right)^2 = E_w \left( \frac{w_1^{(n)}}{m} - \frac{1}{n} \right)^2 = \frac{(1 - \frac{1}{n})}{nm}.$$

We now employ the Chebyshev inequality to bound the  $P_w(\cdot)$  statement of

(39) as follows.

$$\begin{aligned}
& P_w(|\frac{m}{(1-\frac{1}{n})} \sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2 - 1| > \delta) \\
& \leq \frac{m^2}{\delta^2(1-\frac{1}{n})^2} E_w \left( \sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2 - \frac{(1-\frac{1}{n})^2}{m} \right)^2 \\
& = \frac{m^2}{\delta^2(1-\frac{1}{n})^2} \left\{ E_w \left( \sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2 \right)^2 - \frac{(1-\frac{1}{n})^2}{m_n^2} \right\} \\
& = \frac{m_n^2}{\delta^2(1-\frac{1}{n})^2} \left\{ n E_w \left( \frac{w_1^{(n)}}{m} - \frac{1}{n} \right)^4 \right. \\
& \quad \left. + n(n-1) E_w \left( \left( \frac{w_1^{(n)}}{m} - \frac{1}{n} \right)^2 \left( \frac{w_2^{(n)}}{m} - \frac{1}{n} \right)^2 \right) - \frac{(1-\frac{1}{n})^2}{m_n^2} \right\}.
\end{aligned}$$

In view of the fact that  $w_t^{(n)}$ ,  $1 \leq t \leq n$ , have multinomial distribution, after computing  $E_w[(w_1^{(n)})^a (w_2^{(n)})^b]$ , where  $a, b$  are two integers such that  $0 \leq a, b \leq 2$ , followed by some algebra, we can bound the preceding relation above by:

$$\begin{aligned}
& \frac{m^2}{\delta^2(1-\frac{1}{n})^2} \left\{ \frac{(1-\frac{1}{n})}{n^3 m_n^3} + \frac{(1-\frac{1}{n})^4}{m_n^3} + \frac{(m-1)(1-\frac{1}{n})^2}{nm^3} + \frac{4(n-1)}{n^3 m} \right. \\
& \quad \left. + \frac{1}{m^2} - \frac{1}{nm^2} + \frac{n-1}{n^3 m^2} + \frac{4(n-1)}{n^2 m^3} - \frac{(1-\frac{1}{n})^2}{m_n^2} \right\} \\
& \sim \frac{1}{\delta^2} \left\{ \frac{4m}{n^2} + \frac{1}{n^3 m} + \frac{1}{m} + \frac{1}{n^2} + \frac{4}{nm} \right\},
\end{aligned}$$

where  $a_n \sim b_n$  stands for the asymptotic equivalence of the numerical sequences  $a_n$  and  $b_n$ .

Clearly, as  $n, m \rightarrow \infty$ , the preceding relation approaches zero when  $m = o(n^2)$ . Now the proof of (39) is complete.

In view of (39), equivalently to (37), we show that as  $n, m \rightarrow +\infty$  so that  $m = o(n^2)$ , then

$$P_w \{ P_{X|w}(m^{1/2} | G_{n,m}| > \delta_1) > \delta_2 \} \rightarrow 0. \quad (40)$$

In order to establish the preceding conclusion, we first note that  $G_{n,m}$  is a weighted  $u$ -statistic and its kernel  $g(X_i, X_j) := h(X_i, X_j) - \tilde{h}(X_i) - \tilde{h}(X_j)$



possesses the property of (complete) degeneracy (cf. Nasari [15] and Remark 1 therein). On noting now that part (b) of Lemma 2 of Nasari [15] remains true even when the weights are not necessarily non-negative, we use it in the second line of the following argument to conclude

$$\begin{aligned} & P_w\{P_{X|w}(m^{1/2}|G_{n,m}| > \delta_1) > \delta_2\} \\ & \leq P_w\left\{m \sum_{1 \leq i \neq j \leq n} \left(\frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)}\right)^2 > \frac{\delta_1^2 \delta_2}{A E_X h^2(X_1, X_2)}\right\}, \end{aligned} \quad (41)$$

where  $A$  is a positive constant that does not depend on  $n$  or  $m$ . Letting now  $\delta_3^{-1} = \frac{A E_X h^2(X_1, X_2)}{\delta_1^2 \delta_2}$ , we can bound the R.H.S. of the preceding inequality by

$$\begin{aligned} & \delta_3^{-1} m n (n-1) \left\{ E_w \left( \frac{w_1^{(n)} w_2^{(n)}}{m(m-1)} \right)^2 - \frac{2 E_w(w_1^{(n)} w_2^{(n)})}{n m (n-1) (m-1)} + \frac{1}{n^2 (n-1)^2} \right\} \\ & = \delta_3^{-1} m n (n-1) \left\{ \frac{m(m-1)(m-2)(m-3)}{m^2(m-1)^2 n^3(n-1)} + \frac{m(m-1)(m-2)}{m^2(m-1)^2 n^2(n-1)} \right. \\ & \quad \left. + \frac{m(m-1)}{m^2(m-1)^2 n(n-1)} - \frac{2m(m-1)}{m(m-1)n^3(n-1)} + \frac{1}{n^4} \right\} \\ & \sim \delta_3^{-1} \left( \frac{2}{n} + \frac{1}{m} \right) \rightarrow 0, \text{ as } n, m \rightarrow +\infty. \end{aligned}$$

Now the proof of (37) is complete.

To derive (38), we first write  $H_{n,m}$  as follows.

$$\begin{aligned} H_{n,m} &= \sum_{i=1}^n \tilde{h}(X_i) \left\{ \sum_{j=1}^n \left( \frac{w_i^{(n)} w_j^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) - \left( \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n(n-1)} \right) \right\} \\ &= \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m-1} - \frac{1}{n-1} \right) \tilde{h}(X_i) + \sum_{i=1}^n \left( \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n(n-1)} \right) \tilde{h}(X_i) \\ &=: H_{n,m}(1) + H_{n,m}(2). \end{aligned}$$

Note that for  $H_{n,m}(1)$  we have

$$\begin{aligned} H_{n,m}(1) &= \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m} - \frac{1}{n} \right) \tilde{h}(X_i) \\ &\quad + \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) \tilde{h}(X_i). \end{aligned}$$

Since,  $\tilde{h}(X_1), \tilde{h}(X_2), \dots$ , is a sequence of centered i.i.d. random variables, from conclusion (2.13) of Corollary 2.1 of Csörgő *et al.* [8], on replacing the sample variance  $S_n$  by the population variance  $\sigma$  therein, we know that, as  $m, n \rightarrow +\infty$  such that  $m = o(n^2)$ , then

$$P_{X|w} \left( \frac{\sum_{i=1}^n \left( \frac{w_i^{(n)}}{m} - \frac{1}{n} \right) \tilde{h}(X_i)}{\sigma \sqrt{\sum_{i=1}^n \left( \frac{w_i^{(n)}}{m} - \frac{1}{n} \right)^2}} \leq t \right) \rightarrow \Phi(t) \text{ in probability} - P_w$$

for all  $t \in \mathbb{R}$ . By this, in view of (39), to complete the proof of part (a) it only remains to show that

$$P_{X|w}(m^{1/2} \left| \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) \tilde{h}(X_i) \right| > \delta) = o_{P_w}(1), \quad (42)$$

and

$$P_{X|w}(m^{1/2} |H_{n,m}(2)| > \delta) = o_{P_w}(1), \quad (43)$$

where  $\delta > 0$  is an arbitrary positive constant.

To establish relation (42), for  $\delta, \delta_2 > 0$ , we write

$$\begin{aligned} & P_w \left\{ P_{X|w} \left( m^{1/2} \left| \sum_{i=1}^n \left( \frac{w_i^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right) \tilde{h}(X_i) \right| > \delta_1 \right) > \delta_2 \right\} \\ & \leq P_w \left\{ m^{1/2} \sum_{i=1}^n \left| \frac{w_i^{(n)}}{m(m-1)} - \frac{1}{n(n-1)} \right| > \frac{\delta_1 \delta_2}{2E_X |h(X_1, X_2)|} \right\} \\ & \leq \frac{2E_X |h(X_1, X_2)| m^{1/2} n}{\delta_1 \delta_2} \left\{ \frac{m}{m(m-1)n} + \frac{1}{n(n-1)} \right\} \\ & \sim \frac{2E_X |h(X_1, X_2)|}{\delta_1 \delta_2} \left\{ \frac{1}{\sqrt{m}} + \frac{\sqrt{m}}{n} \right\} \rightarrow 0, \end{aligned}$$

as  $m, n \rightarrow +\infty$  such that  $m = o(n^2)$ .

To prove (43), with a similar argument as the one used to prove (42), we bound its  $P_{X|w}(\cdot)$  statement by

$$\begin{aligned} & P_w \left\{ m^{1/2} \sum_{i=1}^n \left| \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n(n-1)} \right| > \frac{\delta_1 \delta_2}{E_X |h(X_1, X_2)|} \right\} \\ & \leq \frac{2E_X |h(X_1, X_2)| m^{1/2} n}{\delta_1 \delta_2} \left( E_w \left( \frac{(w_i^{(n)})^2}{m(m-1)} \right) + \frac{1}{n(n-1)} \right) \\ & \sim \frac{2E_X |h(X_1, X_2)|}{\delta_1 \delta_2} \left( \frac{1}{\sqrt{m}} + \frac{2\sqrt{m}}{n} \right) \rightarrow 0. \end{aligned}$$

This completes the proof of (43) and that of part (a) of this theorem.

The proof of part (b) to a large extent relies on part (a) in view of the following relation.

$$\frac{V_{n,m}^* - V_n}{2\hat{\sigma}_n \sqrt{\sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2}} = \frac{U_{n,m}^* - U_n}{2\hat{\sigma}_n \sqrt{\sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2}} \quad (44)$$

$$+ \frac{\sum_{1 \leq i \neq j \leq n} h(X_i, X_j)}{2\hat{\sigma}_n n^2 (n-1) \sqrt{\sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2}} \quad (45)$$

$$+ \frac{\sum_{i=1}^n (\frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n^2}) h(X_i, X_i)}{2\hat{\sigma}_n \sqrt{\sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2}}. \quad (46)$$

In part (a) it was shown that the conditional distribution of the (44) part of the above equality converges to normal in probability- $P_w$ . We now show the hierarchical asymptotic negligibility of parts (45) and (46). To do so, we first replace the estimator  $\hat{\sigma}_n^2$  by  $\sigma^2 = E_X \tilde{h}^2(X_1)$  and  $\sum_{t=1}^n (\frac{w_t^{(n)}}{m} - \frac{1}{n})^2$  by  $1/m$ .

To deal with (45), it suffices to observe the asymptotic a.s.- $P_X$  negligibility of

$$\frac{m^{1/2}}{n} \cdot \frac{\sum_{1 \leq i \neq j \leq n} h(X_i, X_j)}{n(n-1)}$$

that results from the strong law of large numbers for  $u$ -statistics (cf. for example Serfling (1980)), provided that  $m = o(n^2)$ .

We now use a similar argument to the one used to prove (43) to write

$$\begin{aligned} & P_w \{ P_{X|w}(m^{1/2} | \sum_{i=1}^n (\frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n^2}) h(X_i, X_i)) > \delta_1 \} > \delta_2 \} \\ & \leq P_w \{ m^{1/2} \sum_{i=1}^n | \frac{(w_i^{(n)})^2}{m(m-1)} - \frac{1}{n(n-1)} | > \frac{\delta_1 \delta_2}{E_X |h(X_1, X_1)|} \} \\ & \rightarrow 0, \text{ as } n, m \rightarrow +\infty. \end{aligned}$$

The preceding yields the asymptotic hierarchical negligibility of (46). By this the proof of part (b) and that of Theorem 4.1 are complete.  $\square$

## Appendix

The use of the Hoeffding decomposition to reduce the underlying weighted  $u$ -statistic of order  $p \geq 3$  to weighted partial sums yields  $p - 1$  *completely degenerate* weighted  $u$ -statistics of order  $p$  to order 2. In view of Part (b) of Lemma 2 of Nasari [15], the hierarchical asymptotic negligibility of each one of these completely degenerate weighted  $u$ -statistics can be done by establishing an approximation similar to (41). This should be followed by an application of Chebyshev's inequality, similarly to the argument that follows (41). Similarly to the case  $p = 2$ , the conditional asymptotic normality for the weighted partial sums of i.i.d. random variables, resulting from the Hoeffding reduction, is concluded from Corollary 2.1 of Csörgő *et al.* [8], via steps similar to those used when dealing with  $H_{n,m}$  in Theorem 4.1.  $\square$

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